Homeworks # 1

due Feb. 27, 2015 (11:59pm Hawaii Time)

1. **Synthesizing Stochastic Logic**

In class, we discussed the paradigm of logical computation on stochastic bit streams It is based on a *stochastic representation* of data: each real-valued number x ($0 \leq x \leq 1$) is represented by a sequence of random bits, each of which has probability x of being one and probability $1-x$ of being zero.

In this paradigm, since we are mapping probabilities to probabilities, we can only implement functions that map the unit interval $[0,1]$ onto the unit interval $[0,1]$. Based on the constructs for multiplication and scaled addition shown in Figures 1 and 2, we can readily implement polynomial functions of a specific form, namely polynomials with non-negative coefficients that sum up to a value no more than one:

$$g(t) = \sum_{i=0}^{n} a_i t^i$$

where, for all $i = 0, \ldots, n$, $a_i \geq 0$ and $\sum_{i=0}^{n} a_i \leq 1$.

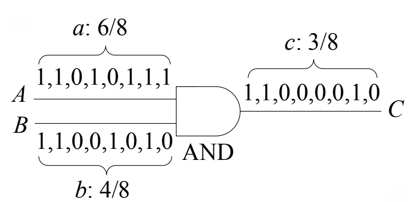


Figure 1: Multiplication on stochastic bit streams with an AND gate. Here the inputs are 6/8 and 4/8. The output is $6/8 \times 4/8 = 3/8$, as expected.

For example, suppose that we want to implement the polynomial $g(t) = 0.3t^2 + 0.3t + 0.2$ through logical computation on stochastic bit streams. We first decompose it in terms of multiplications of the form $a \cdot b$ and scaled additions of the form $sa + (1-s)b$, where s is a constant:

$$g(t) = 0.8(0.75(0.5t^2 + 0.5t) + 0.25 \cdot 1).$$

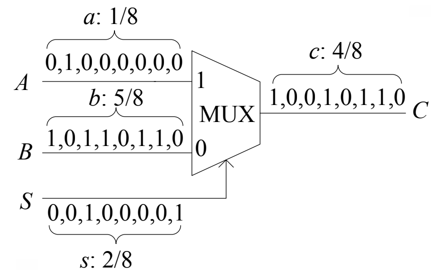


Figure 2: Scaled addition on stochastic bit streams, with a multiplexer (MUX). Here the inputs are $1/8, 5/8$, and $2/8$. The output is $2/8 \times 1/8 + (1 - 2/8) \times 5/8 = 4/8$, as expected.

Then, we reconstruct it with the following sequence of multiplications and scaled additions:

$$\begin{aligned} w_1 &= t \cdot t, \\ w_2 &= 0.5 w_1 + (1 - 0.5)t, \\ w_3 &= 0.75 w_2 + (1 - 0.75) \cdot 1, \\ w_4 &= 0.8 \cdot w_3. \end{aligned}$$

The circuit implementing this sequence of operations is shown in Figure 3. In the figure, the inputs are labeled with the probabilities of the bits of the corresponding stochastic streams. Some of the inputs have fixed probabilities and the others have variable probabilities t. Note that the different lines with the input t are each fed with *independent* stochastic streams with bits that have probability t.

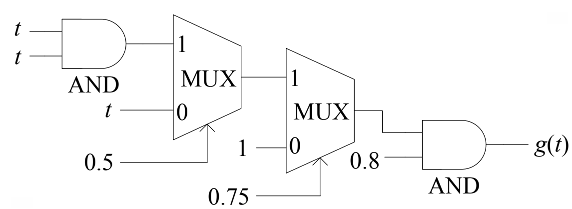


Figure 3: Computation on stochastic bit streams implementing the polynomial $g(t) = 0.3t^2 + 0.3t + 0.2$.

What if the target function is a polynomial that is not decomposable this way? Suppose that it maps the unit interval onto the unit interval but it has some coefficients less than zero or some greater than one. For instance, consider the polynomial $g(t) = \frac{3}{4} - t + \frac{3}{4}t^2$. It is not apparent how to construct a network of stochastic multipliers and adders to implement it.

In class, we discsused a general method for synthesizing arbitrary univariate polynomial functions on stochastic bit streams. A necessary condition is that the target polynomial maps the unit interval onto the unit interval. Our major contribution is to show that this condition is also sufficient: we provide a constructive method for implementing any polynomial that satisfies this condition. Our method is based on some novel mathematics for manipulating polynomials in a special form called a Bernstein polynomial.

We illustrate the basic steps of our synthesis method with the example of $g(t) = \frac{3}{4} - t + \frac{3}{4}t^2$.

(a) Convert the polynomial into a Bernstein polynomial with all coefficients in the unit interval:

$$g(t) = \frac{3}{4} \cdot [(1-t)^2] + \frac{1}{4} \cdot [2t(1-t)] + \frac{1}{2} \cdot [t^2].$$

Note that the coefficients of the Bernstein polynomial are $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$, all of which are in the unit interval.

(b) Implement the Bernstein polynomial with a multiplexing circuit, as shown in Figure 4. The block labeled "+" counts the number of ones among its two inputs; this is either 0, 1, or 2. The multiplexer selects one of its three inputs as its output according to this value. Note that the inputs with probability t are each fed with *independent* stochastic streams with bits that have probability t.

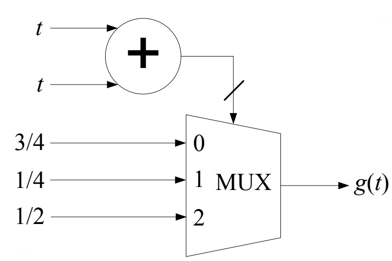


Figure 4: A generalized multiplexing circuit implementing the polynomial $g(t) = \frac{3}{4} - t + \frac{3}{4}t^2$.

Problem

Implement the following polynomial

$$\frac{1}{11}\left(1 - t + 2t^2 - 2t^3 + 3t^4 - 3t^5\right)$$

this way.

Demonstrate how the circuit works on the following input values:

- $X = 0$
- $X = 0.25$
- $X = 0.5$
- $X = 0.75$
- $X = 1$

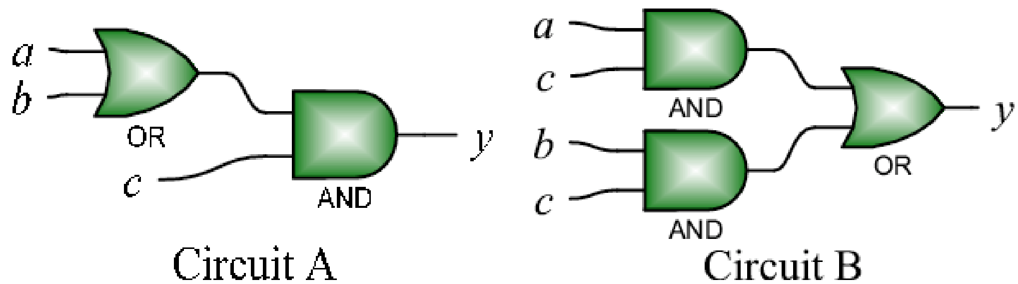
In each case, track probabilities through the circuit that you drew.

2. **Input-Dependent Probability of Failure**

A pervasive problem for digital nano-circuitry is coping with defects and failures. For most nanoscale processes, the devices and components are inherently unreliable. They may exhibit significant variations in their operating parameters. Worse, they may fail intermittently or permanently.

In this problem, you will investigate analysis techniques for digital circuits characterized by logic gates that compute probabilistically: with some probability, each gate produces the incorrect result; with one minus this probability, it produces the correct result.

Consider the two circuits:



Note that both circuits implement the same Boolean function:

$$(a+b)c$$

Suppose that each gate produces the incorrect result (i.e., the complement of the correct Boolean value) with probability ϵ. The probability that the circuits produce the incorrect results are:

$$\textbf{Failure Probability} \begin{cases} P_A = \varepsilon - (2\varepsilon^2 - \varepsilon)c \\ P_B = 3\varepsilon - 5\varepsilon^2 + 2\varepsilon^3 - (\varepsilon - 2\varepsilon^2)(a+b)c + (2\varepsilon^2 - 4\varepsilon^3)abc \end{cases}$$

The failure probability depends on the specific input combination. For some input combinations, Circuit A has a lower probability of failure; for others Circuit B does. The following table gives the failure probabilities for a specific value of ϵ.

$\varepsilon = 0.05$

a	b	c	P_A	P_B	Better
0	0	0	0.050	0.138	*A*
0	0	1	0.095	0.138	*A*
0	1	0	0.050	0.138	*A*
0	1	1	0.095	0.093	*B*
1	0	0	0.050	0.138	*A*
1	0	1	0.095	0.093	*B*
1	1	0	0.050	0.138	*A*
1	1	1	0.095	0.052	*B*

Problem

For the circuit in Figure 5, suppose that each gate produces the incorrect result (i.e., the complement of the correct Boolean value) with the same probability ϵ. For each input assignment of x, y and z compute the probability of obtaining an incorrect result at the outputs c and s.

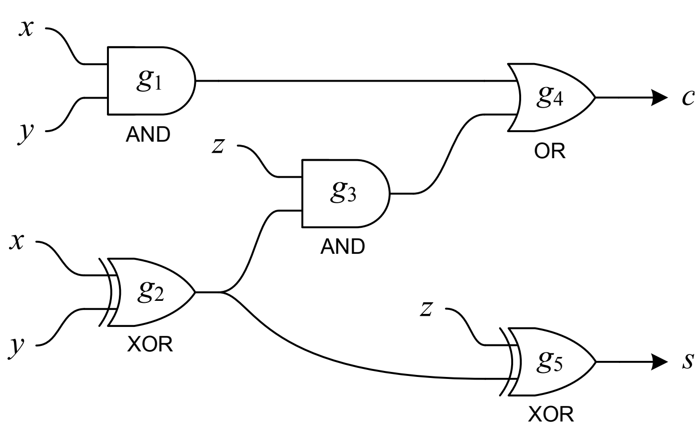


Figure 5: A full adder.

3. **Coding for Lattices of Four-Terminal Switches**

In this problem, we'll consider a new model, applicable to novel technologies such as nanowire crossbar arrays: four-terminal switches. An example is shown in the top part of Figure 6. The four terminals of the switch are all either mutually connected (ON) or disconnected (OFF). We consider networks of four-terminal switches arranged in rectangular *lattices*. An example is shown in the bottom part of Figure 6. Each switch is controlled by a Boolean literal. If the literal takes the value 1 (0) then corresponding switch is ON (OFF). The Boolean function for the lattice evaluates to 1 iff there is a closed path between the top and bottom edges of the lattice. The function is computed by taking the sum of the products of the literals along each path. These products are $x_1x_2x_3$, $x_1x_2x_5x_6$, $x_4x_5x_2x_3$, and $x_4x_5x_6$.

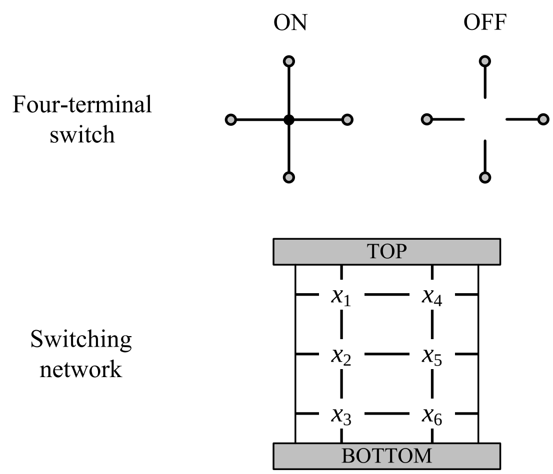


Figure 6: Four-terminal switching network implementing the Boolean function $x_1x_2x_3 + x_1x_2x_5x_6 + x_2x_3x_4x_5 + x_4x_5x_6$.

Problem

Implement the following functions in rectangular lattices of four-terminal switches. There should be a closed path from the top to bottom plates if and only if the Boolean function evaluates to 1. Use the smallest lattice possible. (See " Logic Synthesis for Switching Lattices Mustafa Altun and Marc Riedel, IEEE Transactions on Computers, 2011.)

(a) Function 1:

$$a(b + c(d + e))$$

(b) Function 2:

$$x_1 x_2 x_3 + \bar{x}_1 \bar{x}_2 \bar{x}_3 + x_1 \bar{x}_2 + x_2 \bar{x}_3$$

(c) Function 3:

$$ab + bc + cd + ef$$

(d) Function 4:

$$x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5$$

(e) Function 5:

$$x_1 x_2 \oplus x_2 x_3 \oplus x_3 x_4 \oplus x_4 x_5 \oplus x_5 x_1$$

Here + denotes OR; multiplication represents AND; $\oplus$ denotes Exclusive-OR; a over-bar represents NOT.

4. **Percolation**

Percolation theory is a rich mathematical topic that forms the basis of explanations of physical phenomena such as diffusion and phase changes in materials. It tells us that in media with random local connectivity, there is a critical threshold for global connectivity: below the threshold, the probability of global connectivity quickly drops to zero; above it, the probability quickly rises to one.

Broadbent and Hammersley described percolation with the following metaphorical model. Suppose that water is poured on top of a large porous rock. Will the water find its way through holes in the rock to reach the bottom? We can model the rock as a collection of small regions each of which is either a hole or not a hole. Suppose that each region is a hole with independent probability p_1 and not a hole with probability $1 - p_1$. The theory tells us that if p_1 is above a critical value p_c, the water will always reach the bottom; if p_1 is below p_c, the water will never reach the bottom. The transition in the probability of water reaching bottom as a function of increasing p_1 is extremely abrupt. For an infinite size rock, it is a step function from 0 to 1 at p_c.

In two dimensions, percolation theory can be studied with a lattice, as shown in Figure 7(a). Here each site is black with probability p_1 and white with probability $1-p_1$. Let p_2 be the probability that a connected path of black sites exists between the top and bottom plates. Figure 7(b) shows the relationship between p_1 and p_2 for different square lattice sizes. Percolation theory tells us that with increasing lattice size, the steepness of the curve increases. (In the limit, an infinite lattice produces a perfect step function.) Below the critical probability p_c, p_2 is approximately 0 and above it p_2 is approximately 1.

Suppose that each site of a percolation lattice is a four-terminal switch controlled by the same literal x_1. Also suppose that each switch is independently defective with the same probability. Defective switches are represented by white and black sites while the switch is supposed to be ON and OFF, respectively. Let's analyze the cases $x_1 = 0$ and $x_1 = 1$. If $x_1 = 0$ then each site is black with the defect probability, and the defective black sites might cause an error by forming a path between the top and bottom plates. In this case, p_1 and p_2 described in the percolation model correspond to the defect probability and the probability of an error in top-to-bottom connectivity, respectively. If $x_1 = 1$ then each site is white with the defect probability and the defective white sites might cause an error by destroying the connection between the top and bottom plates. In this case, p_1 and p_2 in the percolation model correspond to $1-$(defect probability) and $1-$(probability of an error in top-to-bottom connectivity), respectively. The

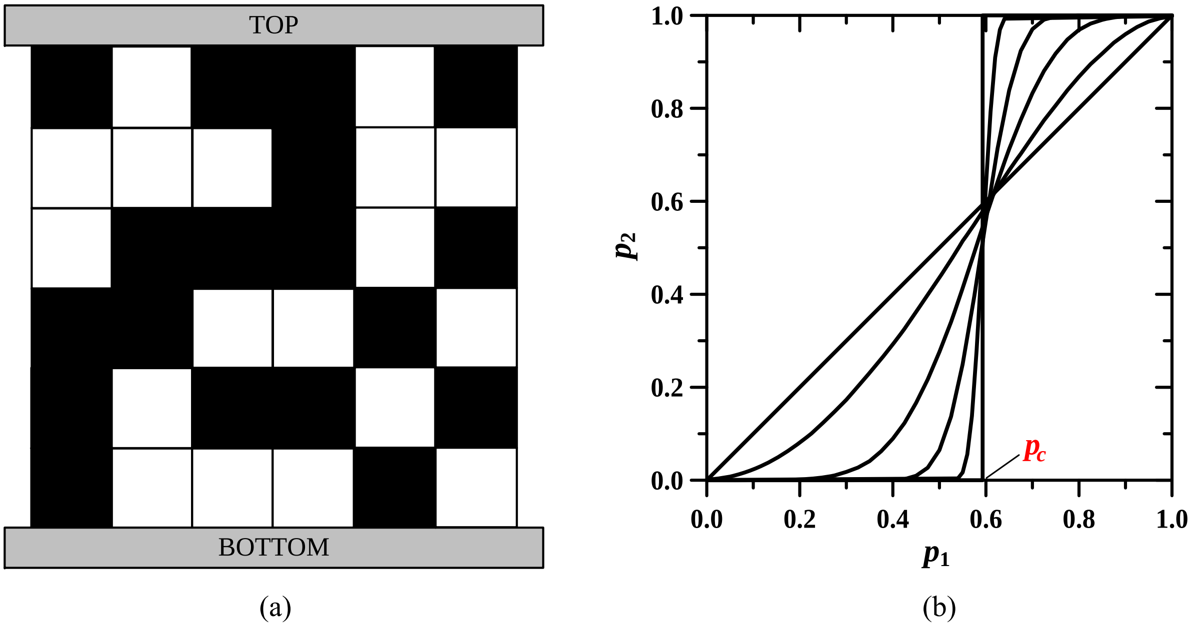


Figure 7: (a): Percolation lattice with random connections; there is a path of black sites between the top and bottom plates. (b) p_2 versus p_1 for 1×1, 2×2, 6×6, 24×24, 120×120, and infinite-size lattices.

relationship between p_1 and p_2 is shown in Figure 8.

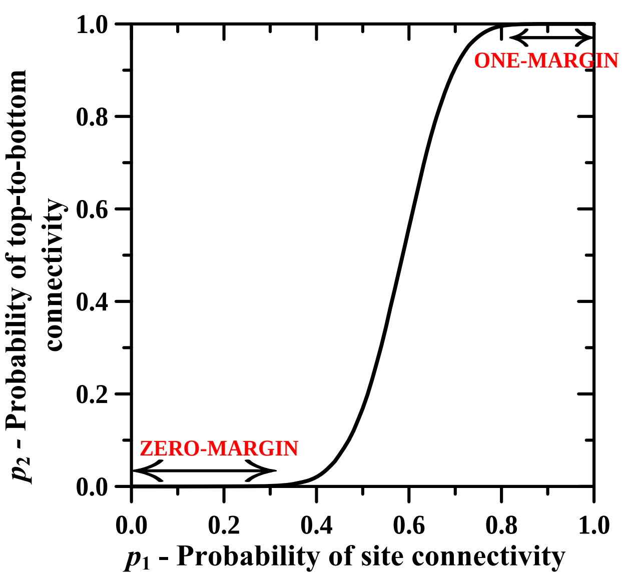


Figure 8: Non-linearity through percolation in random media.

Problem

Compute or estimate the critical thresholds for percolation in the following models.

(a) Suppose you have a graph consisting of n vertices. Each vertex has a collection of m neighbors (randomly chosen from the total set of n vertices). It is connected to each neighbor with an edge with probability p. Consider connectivity between a randomly chosen specific pair of vertices, A and B. Consider the probability that they are connected as a function of n, m and p. What is the critical threshold for percolation?

(b) Consider the following scenario for an ad-hoc peer-to-peer mobile network. There is a cell phone tower located in the center of some geographical area. There are n mobile users located at random locations with a r kilometer radius of the tower. Each user's phone can communicate with another user or with the tower if they are within q kilometers. Consider the probability that every user can communicate (directly or via a sequence of hops through other users) with the tower as a function of n, r and q. What is the critical threshold for percolation?